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SOLUTION SKETCHES TO HOMEWORK 9

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Homework 9.1 (True or false?)**.** Prove the following statements or give a counterexample.

a. The image of a simply connected domain under a nonconstant holomorphic function is again a simply connected domain.
b. The image of a simply connected domain under an injective holomorphic function is again a simply connected domain.
c. The complex plane is not biholomorphically equivalent to any simply connected domain $G \subsetneq \mathbf{C}$.
d. The set $\mathbf{C} \setminus B_r(z_0)$, where $z_0 \in \mathbf{C}$ and $r > 0$, is simply connected.

Solution. a. *False.* The complex exponential maps the simply connected set $\mathbf{C}$ to $\mathbf{C} \setminus \{0\}$, which is not simply connected.

b. *True.* The open mapping theorem implies $f(G)$ is a domain. Lemma 6.1 shows the inverse $f^{-1}\colon f(G) \to G$ is holomorphic, so in particular continuous. Let γ be a continuous closed curve through $f(G)$. Then $f^{-1} \circ \gamma$ is a continuous closed curve in G. Since G is simply connected, there exist a continuous map $H\colon [0,1]^2 \to G$ and $z_0 \in G$ such that $H(0,t) = f^{-1} \circ \gamma(t)$, $H(1,t) = z_0$ and $H(s,0) = H(s,1)$ for every $s,t \in [0,1]$. The continuous map $S := f \circ H$ then obeys $S(0,t) = \gamma(t)$, $S(1,t) = f(z_0)$ and $S(s,0) = S(s,1)$ for every $s,t \in [0,1]$. Hence γ can be contracted to a point.

c. *True.* If it were biholomorphically equivalent to a simply connected domain $G \subsetneq \mathbf{C}$, then by the Riemann mapping theorem it would also be biholomorphically equivalent to the unit disk $B_1(0)$, which contradicts Liouville's theorem.

d. *False.* Otherwise the assignment $z \mapsto (z - z_0)^{-1}$ would have a primitive on the set $\mathbf{C} \setminus B_r(z_0)$. But this contradicts the fact that $\int_{\partial B_{2r}(z_0)} (z - z_0)^{-1}\,\mathrm{d}z \neq 0$.

Homework 9.2 (Schwarz lemma on simply connected domains)**.** Let $G \subsetneq \mathbf{C}$ be a simply connected domain. Given $a \in G$ we denote by $\mathrm{Hol}_a(G)$ the set of holomorphic functions $f\colon G \to G$ such that $f(a) = a$. Show $|f'(a)| \leq 1$ for every $f \in \mathrm{Hol}_a(G)$. Moreover, show $f \in \mathrm{Hol}_a(G)$ is bijective if and only if $|f'(a)| = 1$[1].

Solution. Let $g\colon G \to B_1(0)$ be the biholomorphic map given by the Riemann mapping theorem. Set $z_0 := g(a) \in B_1(0)$ and, as usual, we will denote by $\varphi_{z_0} : B_1(0) \to B_1(0)$ the following biholomorphic map introduced in Homework 7.2:

$$\varphi_{z_0}(z) := \frac{z - z_0}{1 - \overline{z_0}\, z}.$$

Upon replacing g with $\varphi_{z_0} \circ g$ we may and will assume $g(a) = 0$. Then $h := g \circ f \circ g^{-1}$ is holomorphic and satisfies $h(0) = 0$. Hence by the standard Schwarz lemma we deduce $|h'(0)| \leq 1$. Using the chain rule, we find

$$|g'(a)||f'(a)||(g^{-1})'(0)| = |g'(f(g^{-1}(0)))f'(g^{-1}(0))(g^{-1})'(0)| \leq 1.$$

Since $(g^{-1})'(0) = 1/g'(a) \neq 0$, we infer $|f'(a)| \leq 1$, as claimed. If, on the one hand, f is bijective, then the Schwarz lemma (applied to h and its inverse) implies that h is a rotation, so that in the above estimate we have equality. As before, we conclude $|f'(a)| = 1$. On

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[1]**Hint.** Use the function given by the Riemann mapping theorem.

the other hand, if $|f'(a)| = 1$, the chain rule implies again that $|h'(0)| = 1$, so that h is a rotation. Hence f has to be bijective.

Homework 9.3 (Singularities of injective holomorphic maps)**.** Let $f\colon U \setminus \{z_0\} \to \mathbf{C}$ be holomorphic and injective, where $U \subset \mathbf{C}$ is open. Prove either z_0 is a removable singularity and the continuous extension to z_0 is still injective or z_0 is a pole of first order[2].

Solution. Assume z_0 is a removable singularity. By contradiction, we suppose the extension of f is not injective. This means there exists $z_1 \in U \setminus \{z_0\}$ such that $f(z_0) = f(z_1)$. Let $r > 0$ be such that $B_r(z_0) \cap B_r(z_1) = \emptyset$ and $B_r(z_0), B_r(z_1) \subset U$. Since the extension is still holomorphic on U we know (by the open mapping theorem) that $f(B_r(z_0))$ and $f(B_r(z_1))$ are open, since the function f is not constant. Thus their intersection is also open and nonempty since $f(z_0) \in f(B_r(z_0)) \cap f(B_r(z_1))$. This means the intersection contains more than one element. This contradicts the injectivity of f on $U \setminus \{z_0\}$.

Next we assume that z_0 is not a removable singularity. Again by contradiction, let us suppose it is an essential singularity. Then for every sufficiently small radius $r > 0$, the set $f(B_r(z_0) \setminus \{z_0\})$ is open (again by the open mapping theorem). Moreover, also the set $f(U \setminus \bar{B}_r(z_0))$ is open and nonempty. Due to injectivity, they are disjoint. Hence $f(B_r(z_0) \setminus \{z_0\})$ cannot even be dense in $\mathbf{C}$, contradicting the Picard's great theorem (or the weaker Casorati–Weierstraß theorem). We conclude f has a pole in z_0. This implies there exists a ball $B_s(z_0)$ such that $f(z) \neq 0$ for every $z \in B_s(z_0) \setminus \{z_0\}$. Hence the function $g := 1/f$ is holomorphic, injective and has a removable singularity in z_0. As in the first paragraph, we can extend it to a holomorphic and injective function $g\colon B_s(0) \to \mathbf{C}$ such that $g(z_0) = 0$. Then Lemma 6.1 yields $g'(z_0) \neq 0$, which means g has a first order zero in z_0. Consequently, f has a first order pole in z_0.

Homework 9.4 (Rigidity of biholomorphic maps*)**.**
a. Let $f\colon B_1(0) \to B_1(0)$ be a biholomorphic map which satisfies $f(a) = a$ and $f(b) = b$ for two distinct points $a, b \in B_1(0)$. Show f is the identity map on $B_1(0)$.
b. Show $f\colon \mathbf{C} \to \mathbf{C}$ is holomorphic and injective if and only if f is of the form $f(z) = az + b$ for some $a \in \mathbf{C} \setminus \{0\}$ and $b \in \mathbf{C}$[3].
c. Show $f\colon \mathbf{C} \setminus \{0\} \to \mathbf{C} \setminus \{0\}$ is holomorphic and injective if and only if f is of the form $f(z) = az$ or $f(z) = a/z$ for some $a \in \mathbf{C} \setminus \{0\}$.
d. Let $G \subsetneq \mathbf{C}$ be a simply connected domain and let $f \in \mathrm{Hol}_a(G)$ be biholomorphic, where $a \in G$ (cf. Homework 9.2). Show $f'(a) \in (0, \infty)$ implies f is the identity map on G.

Homework 9.5 (Examples of the Riemann mapping theorem)**.** In this exercise we build biholomorphic maps $f\colon G \to B_1(0)$ for some special sets $G \subset \mathbf{C}$.
a. Show the assignment $z \mapsto (z - \mathrm{i})(z + \mathrm{i})^{-1}$ is biholomorphic from the upper half-plane $\mathbf{H}_+ = \{z \in \mathbf{C} : \Im z > 0\}$ to $B_1(0)$.
b. Find a biholomorphic map $f\colon \mathbf{C} \setminus (-\infty, 0] \to B_1(0)$.

Solution. a. Clearly, the map is well-defined as $-\mathrm{i} \notin \mathbf{H}_+$. Moreover, we have

$$|z - \mathrm{i}| < |z + \mathrm{i}| \quad \Longleftrightarrow \quad z\mathrm{i} - \bar{z}\mathrm{i} < -z\mathrm{i} + \bar{z}\mathrm{i} \quad \Longleftrightarrow \quad \Im z > 0.$$

It remains to show the map is bijective. This is done by giving the inverse explicitly, which is easily checked to be the assignment $z \mapsto \mathrm{i}(z + 1)(1 - z)^{-1}$.

b. Since $G := \mathbf{C} \setminus (-\infty, 0]$ is star-shaped, it is simply connected. Hence there exists a holomorphic square-root defined on G. It is injective; we claim $\Re\sqrt{z} > 0$ for all $z \in G$. Indeed, taking the principal branch of the logarithm such that $\arg\log(z) \in (\pi, \pi)$ for all $z \in G$, the square root can be written as $\sqrt{z} = \exp(\log(z)/2)$, thus $\arg\sqrt{z} \in (-\pi/2, \pi/2)$. Moreover, for any $z \in \mathbf{C}$ with $\Re z > 0$ we know $\arg z \in (-\pi/2, \pi/2)$, so that $z^2 \in G$.

[2]**Hint.** Rule out an essential singularity using Picard's great theorem.

[3]**Hint.** Apply Homework 9.3.

Consequently, the map $\sqrt{\cdot}\colon G \to \{z \in \mathbf{C} : \Re z > 0\}$ is biholomorphic. As a next step we note that by the rotation $z \mapsto \mathrm{i}z$ we can map $\{z \in \mathbf{C} : \Re z > 0\}$ biholomorphically to $\mathbb{H}_+$. (Recall multiplication with i represents a counterclockwise rotation by 90 degrees.) We use a. to map $\mathbb{H}_1$ biholomorphically to $B_1(0)$. The composition reads

$$f(z) = \frac{\mathrm{i}\sqrt{z} - \mathrm{i}}{\mathrm{i}\sqrt{z} + \mathrm{i}} = \frac{\sqrt{z} - 1}{\sqrt{z} + 1}.$$